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The computed characteristics of turbulent flow and convection in concentric circular annuli. Part II. Uniform heating on the inner surface

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Abstract

Exact numerical solutions, essentially free of empiricism were obtained for fully developed turbulent forced convention in concentric circular annuli with uniform heating on the inner wall and no heat transfer the through the outer wall. The numerically computed values of *Nu* are represented almost exactly as a function of Nu_0, Nu_1, Nu_∞, and Pr_t/Pr. Here, Nu_0 and Nu_∞ are the limiting and asymptotic solutions for $Pr = 0$ and $Pr \to \infty$ respectively, and Nu_1 is the special solution for $Pr = Pr_t \approx 0.8673$. The predicted values for all *Re*, all *Pr*, and all aspect ratios are in agreement with the experimental data within their scatter.

1. Introduction

Double-pipe heat exchangers, in which one fluid passes through an inner round tube and a second fluid through the annulus formed by a concentric outer round tube, are utilized extensively in industrial processing. Such heat exchange invokes many parameters, including the aspect ratio a_1/a_2, and, for single-phase fluids, the relative direction of flow of the two streams (concurrent or countercurrent), the two mass rates of flow, and two sets of physical properties such as k, c, μ, and ρ. For the transfer of sensible heat between countercurrent streams of the same fluid at the same enthalpic rate, a uniform heat flux density occurs over the dividing surface insofar as changes in the physical properties with temperature can be neglected and insofar as the exchanger is of sufficient length so that end-effects (developing flow and/or developing convection) can also be neglected. This thermal boundary condition may also be established, at least approximately, by longitudinal electrical heating of an solid axial core. On the other hand, one of the fluid streams may condense or boil. Because of the large heat transfer coefficients for boiling and condensing, an

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Nomenclature

a_1	inner radius of annulus (m)
a_2	outer radius of annulus (m)
a_0	radius of zero shear stress (m)
a_{max}	radius of maximum in velocity (m)
a^+	dimensionless radius $[a(\tau_w\rho)^{1/2}/\mu]$
c	specific heat capacity (J/kg K)
D	diameter (m)
f	Fanning friction factor $[2\tau_w/\rho u_m^2]$
h	heat transfer coefficient (W/m^2 K)
k_t	eddy conductivity (W/m K)
k	thermal conductivity (W/m K)
j	radial heat flux density (W/m^2)
Nu	Nusselt number $[2h(a_2 - a_1)/k]$
Nu_0	$Nu\{Pr = 0\}$
Nu_1	$Nu\{Pr = Pr_t\}$
Nu_∞	$Nu\{Pr \to \infty\}$
P	pressure (Pa)
Pr	Prandtl number $[c\mu/k]$
Pr_t	turbulent Prandtl number $\left[\frac{Pr(\overline{u'v'})^{++}\left(1-(\overline{T'v'})^{++}\right)}{(\overline{u'v'})^{++}\left(1-(\overline{T'v'})^{++}\right)}\right]$
r	radial coordinate (m)
r^+	dimensionless radius $[r(\tau_{w1}\rho)^{1/2}/\mu]$
R	radius ratio $[r/a_1]$
Re	Reynolds number $[2(a_2 - a_1)\rho u_m/\mu]$
T	time-averaged temperature (K)
T^+	dimensionless temperature $[k(\rho\tau_{w1})^{1/2}(T_{w1} - T)/\mu j_{w1}]$
T_m	mixed-mean temperature (K)
T'	fluctuating component of temperature (K)
$\overline{T'v'}$	time-average of product of fluctuating temperature and velocity (K m/s)
$(\overline{T'v'})^{++}$	local fractional of radial heat flux density due to turbulence $[\rho c\overline{T'v'}/j]$
u	axial component of time-averaged velocity (m/s)
u^+	dimensionless axial velocity $[u/(\rho\tau_{w1})^{1/2}]$
u_m	mixed-mean axial velocity (m/s)
u'	fluctuating component of axial velocity (m/s)
$\overline{u'v'}$	time-average of product of fluctuating components of velocity (m^2/s^2)
$(\overline{u'v'})^{++}$	local fraction of shear stress due to turbulence $[-\rho\overline{u'v'}/\tau]$
$(\overline{u'v'})^{+}$	alternative dimensionless shear stress $[-\rho u'v'/\tau_{w1}]$
v	radial component of time-averaged velocity (m/s)
v'	fluctuating component of radial velocity (m/s)
y	distance from wall (m)
y^+	dimensionless distance from wall $[y(\rho\tau_w)^{1/2}/\mu]$
z	axial coordinate (m)
γ	$[(j/j_{w1})(\tau_{w1}/\tau) - 1]$
μ	dynamic viscosity (Pa s)
μ_t	eddy dynamic viscosity (Pa s)
ρ	specific density (kg/m^3)
τ	shear stress (Pa)
τ_w	shear stress at wall (Pa)

Subscripts

w1	based on shear stress on the inner wall
w2	based on shear stress on the outer wall
wm	based on mean shear stress on the walls

essentially uniform temperature is then attained on the surface through which heat is being transferred. Because the postulate of either a uniform heat flux density or a uniform surface temperature greatly simplifies theoretical modeling, and because uniform heating generally constitutes an upper bound and uniform wall temperature a lower bound for the convective heat transfer coefficient, most analyses of heat transfer postulate one or the other of these two thermal boundary conditions even though the required conditions are never wholly fulfilled in practice. Effective thermal insulation is usually placed on the outer surface of a double-pipe heat exchanger to reduce heat losses to the surroundings and/or to protect personnel from extreme temperatures. Accordingly, the postulate of an adiabatic external surface is usually a reasonable one. This latter conclusion has been also been reached by prior analysts and experimentalists.

In the current investigation, a uniform heat flux density on the inner surface, perfect insulation on the outer surface, sensible transfer heat only, fully developed turbulent flow, fully developed one-dimensional convection, and invariant physical properties are postulated. The latter two conditions are difficult to establish experimentally. Positioning of the inner tube or core along the axis, particularly for very small aspect ratios, without introducing entrance and exit effects in the flow is a real challenge experimentally, while even a slight misalignment or mislocation perturbs the velocity distribution and may result in a secondary motion. These disturbances of the flow by the supports for the central tube or core as well as those due to its misalignment and/or mislocation may influence the value of the heat transfer coefficient significantly. The large temperature differences that are necessary if the heat transfer coefficient is to be determined with accuracy, but may result in significant radial variations in the viscosity, thermal conductivity, and density, which in turn perturb the value of the heat transfer coefficient. These physical property effects are a function of the temperature distribution in the fluid near the wall and are not uniquely characterized by the temperature-difference between the wall and the bulk of the fluid. They are also fluid-specific and depend on whether the fluid is being heated or cooled.

Because of the great industrial importance of double-pipe heat exchangers, one might expect to find in the archival literature many theoretical analyses, many sets of experimental data, and many correlating equations for heat transfer in annuli. This expectation is fulfilled to a degree, but most of the sets of relevant experimental data that were identified in the course of this investigation are quite old, quite limited in scope, and, as indicated by their scatter, of poor precision and accuracy. The secondary effects mentioned in the previous paragraph have rarely been investigated in a controlled manner. Many of the sets of measurements are given only in graphical form or in compound forms such as $Nu/RePr^{1/3}f^{1/2}$ without separate specification of Re, Pr, and f, thereby making the determination of Nu in future studies such as the current one uncertain in a numerical sense. Leung [1] presented a thorough and very discerning review of the experimental data that preceded his own work, precluding the need for detailed discussion of that work here, while more recently Childs and Long [2] reviewed the surprisingly few subsequent experimental and theoretical investigations. Despite its age, by far the most reliable numerical analysis of broad scope appears to be that of Leung (also reported in [3]). The other analyses, with one exception, need not be mentioned herein. Most of the prior numerical analyses, including that of Leung, although generally sound in their basic formulation, are subject to considerable error because of the utilization of the eddy viscosity to represent the time-averaged turbulent shear stress, and in most cases, incoherent and inaccurate expressions for that heuristic quantity and for the time-averaged velocity distribution. The eddy diffusivity is invalid in a fundamental sense in annular flow. (See, for example, [4].) Instead of the eddy diffusivity, the analysis of Wilson and Medwell [5] utilized a mixing-length formulation, which is also invalid in a fundamental sense. Furthermore, they introduced an erroneous functional dependence for the velocity and thereby for the mixing-length on distance from the nearest wall. Many of the prior numerical analyses, although not those of Leung and of Wilson and Medwell, are subject to further uncertainty because of the postulate of an overly idealized total heat flux density distribution. *Direct numerical simulations* (DNS) are free of these fundamental sources of error, but they are at present limited to rates of flow barely above the minimum for the attainment of fully developed turbulence. The correlative and predictive equations in the literature for Nu are almost all based on the adaptation of a prior power-law-type expression for round tubes simply by introduction of the hydraulic diameter as the characteristic dimension.

The numerical calculations of convection in this investigation depend critically upon the spatial distribution and parametric dependences of both the turbulent shear stress and the total shear stress. The time-averaged velocity distribution and the space-mean velocity also serve as nominal inputs but their values are fixed unambiguously, both functionally and numerically, by the distribution of the turbulent shear stress. The thermal results herein are presumed to constitute an improvement over prior ones primarily by virtue of the use of essentially exact values for the radial distribution of the total heat flux density and for the fractions of the total shear stress and the heat flux density due to turbulence. The merit of the direct use of the dimensionless turbulent shear stress as a variable for the description and prediction of flow in an annulus is amply demonstrated by Kaneda et al. [6] in Part I of this investigation and will not be belabored here. The uncertainty in the results for flow due to the use of empirical expressions for the location of the maximum in the velocity distribution and the zero in the total shear stress distribution carries over to the thermal computations but is not presumed to be significant in either case. Only those expressions for flow that are directly utilized for the calculation of convection are reproduced here.

For any chosen set of thermal boundary conditions, one additional independent parameter arises in the thermal calculations relative to those for flow, namely the Prandtl number, $Pr = c\mu/k$, as well as one dependent parameter, namely the turbulent Prandtl number, Pr_t. This latter quantity, which was originally defined in terms of the eddy viscosity μ_t, and the eddy conductivity k_t, is redefined herein in terms of non-heuristic variables but it remains a source of uncertainty.

Computed values of Nu for turbulent flow in annuli for the less important cases of uniform heating or cooling on the outer wall, for combined heating or cooling on both walls, and for heating or cooling with uniform wall temperature(s) will be presented in Part III. Finally, generalized algebraic predictive equations for all of these thermal boundary conditions and all values of Re, Pr, and a_1/a_2, including the limiting cases of round tubes and parallel- plate channels will be presented in Part IV.

2. Formulations for flow

The time-averaged and once integrated differential momentum balance for steady fully developed flow of a fluid with invariant physical properties in a circular concentric annulus may be expressed as

$$\tau = -\mu\left(\frac{du}{dr}\right) - \rho(\overline{u'v'}). \tag{1}$$

Here, τ is the total time-averaged shear stress in the z (axial) direction imposed on the fluid at r, the radial distance from the axis by the fluid at greater values of r; μ and ρ are the dynamic viscosity and specific density of the fluid; u is the local time-averaged velocity; and $(\overline{u'v'})$ is the local time-averaged product of the fluctuating components of the velocity in the axial and radial directions. Churchill and Chan [4] proposed that Eq. (1) be re-expressed in the following dimensionless form:

$$\frac{\tau}{\tau_{w1}} = \frac{du^+}{dy^+} + (\overline{u'v'})^+. \tag{2}$$

Here, τ_{w1} is the shear stress in the fluid at the inner wall, $u^+ = u(\rho/\tau_{w1})^{1/2}$ and $y^+ = y(\rho\tau_{w1})^{1/2}/\mu$ are the classical "wall-variables" of Prandtl, and $(\overline{u'v'})^+ = -\rho(\overline{u'v'})/\tau_{w1}$ is their analog for the local turbulent shear stress. This latter dimensionless variable may be interpreted physically as the local turbulent shear stress as a fraction of the shear stress at the wall. Eq. (2), as well as Eq. (1), is exact within the afore-mentioned restrictions on the flow and fluid.

Rather than following the traditional path, which consists of introducing a heuristic expression such as the eddy viscosity to represent the local turbulent shear stress in Eq. (2), and devising a correlating equation for that quantity, Churchill and Chan [7] devised a theoretically structured correlating equation for $(\overline{u'v'})^+$ itself, which Heng et al. [8] subsequently up-dated numerically on the basis of the new improved experimental data of Zagarola [9] for $u\{y^+\}$ and u_m^+ in a round tube. The concept of Churchill and Chan [4] of utilizing the dimensionless turbulent shear stress as a variable rather than introducing some heuristic quantity such as the eddy viscosity proved to be even more advantageous than expected in that the common correlating equation for $(\overline{u'v'})^+$ for flow in round tubes and between parallel plates was found to be simpler than the equivalent one for the eddy viscosity. Furthermore, as a direct consequence of the use of $(\overline{u'v'})^+$ as a variable, they discovered that the mixing length is singular in all channels, and also confirmed on new theoretical grounds the validity of the assertion of Maubach and Rehme [10] that the eddy viscosity is singular at one point and negative over an adjacent region in all channels (such as annuli) for which the velocity distribution is not symmetrical or anti-symmetrical. It follows that all solutions for flow in annuli based wholly on the eddy viscosity, are fundamentally unsound and subject to functional as well as numerical inaccuracies on that basis. *Large eddy simulation* (LES) appears to be valid for annuli insofar as the eddy diffusivity is not utilized near $r = a_0$, but the use of "wall functions" is inferior numerically and functionally to the direct use of the turbulent shear stress in the buffer and viscous sublayers. Churchill [11] subsequently proposed the use of $(\overline{u'v'})^{++} = -\rho(\overline{u'v'})/\tau$, which may be interpreted as the local fraction of the shear stress due to turbulence, rather than $(\overline{u'v'})^+$. This alternative dimensionless variable was found to simplify the process of carrying out numerical solutions for flow in round tubes and parallel-plate channels. However, in annuli it shares the singularity of the eddy viscosity and the mixing length and is thereby inapplicable.

In view of these several considerations, Kaneda et al. [6] devised separate correlating equations for $(\overline{u'v'})^+$ for the inner and outer regions of annuli (as defined by $r = a_{max}$) on the basis of the correlating equation of Churchill and Chan [7] for the turbulent shear stress in round tubes and parallel-plate channels as up-dated numerically by Heng et al. [8]. The resulting expressions for $(\overline{u'v'})^+$ for the inner and outer regions of annuli are much more complex than the common single one for a round tube and a parallel-plate channel because of the asymmetry of the flow and the related non-linear variation of the total shear stress across the annulus. In the interests of brevity and minimal repetition, the detailed expressions for $(\overline{u'v'})^+$ are not reproduced here. In spite of their relative complexity, these modified expressions proved to be quite successful as an input to the numerical integrations for the velocity distribution and the friction factor for all aspect ratios and for a complete range of the rate of flow above the minimum for fully developed turbulence. The accuracy of the resulting predictions of $u^+\{y^+\}$ and u_m^+ in annuli, as established by comparisons with experimental data, provides confidence in the use of direct adaptations of the correlating equations for $(\overline{u'v'})^+$ for round tubes and parallel-plate channels, along with their counterparts for u^+ and u_m^+, to predict convection in annuli.

3. Exact formulations for convection

The time-averaged and once-integrated differential energy balance for steady fully developed convection in the fully developed turbulent flow of a single-phase fluid with invariant physical properties in a concentric circular annulus may be expressed as

$$j = -k\frac{dT}{dr} + \rho c(\overline{T'v'}). \tag{3}$$

Here j is the total local heat flux density in the radial direction due to both the molecular motion (thermal conduction) and the fluctuating components of the velocity (turbulent transport). It follows from an energy balance over an annular segment of fluid between any radial location r and the outer radius of the annulus a_2 that

$$j = \frac{\rho c}{2r}\int_{r^2}^{a_2^2} u\left(\frac{\partial T}{\partial z}\right) dr^2. \tag{4}$$

Following Churchill [11], Eq. (3) may be expressed in dimensionless form as

$$\frac{j}{j_{w1}}\left[1-(\overline{T'v'})^{++}\right] = \frac{dT^+}{dr^+}, \tag{5}$$

and Eq. (4) as

$$\frac{j}{j_{w1}} = \frac{1}{R[(a_2/a_1)^2-1]}\int_{R^2}^{(a_2/a_1)^2} \frac{u}{u_m}\left(\frac{\partial T/\partial z}{\partial T_m/\partial z}\right) dR^2. \tag{6}$$

Here $j_{w1} = j\{r = a_1\}$, $T^+ = k(\tau_{\omega 1}\rho)^{1/2}(T_{\omega 1} - T)/\mu j_{\omega 1}$, $(\overline{T'v'})^{++} = \rho c(\overline{T'v'})/j$, and $R = r/a_1$. Since $(\overline{T'v'})/j$ remains positive and finite across the annulus, it is convenient, in spite of the concurrent use of $(\overline{u'v'})^+$, to utilize $(\overline{T'v'})^{++}$, the local fraction of the heat flux density due to turbulence, rather than $(\overline{T'v'})^+ = \rho c(\overline{T'v'})/j_{w1}$ as the dependent variable . The representation of the turbulent transport directly in terms of the time-average of the fluctuating value of $T'v'$ rather than in terms of a heuristic quantity such as the eddy conductivity proves to be just as advantageous in predicting heat transfer as did the representation of the turbulent transfer of momentum by $(\overline{u'v'})^{++}$ or $(\overline{u'v'})^+$ rather than by the eddy viscosity in predicting flow.

In spite of the complication imposed by the parametric dependence on Pr, and the lesser data base of experimental values for $\overline{T'v'}$ and T, asymptotic solutions for $(\overline{T'v'})^{++}$ could undoubtedly be derived and utilized to construct a generalized correlating equation that could be utilized with Eq. (5) to determine T^+r^+ and in turn T_m^+, just as was done to predict values of u^+ and u_m^+. However, as shown in Section 7, a quite different and much better procedure was discovered on the basis of (1), the use of Pr_t/Pr rather than $(\overline{T'v'})^{++}$ as the explicit dependent variable, and (2), the use of a formal analogy between momentum transfer and energy transfer that, while not exact, is nevertheless free of empiricism.

In order to replace $(\overline{T'v'})^{++}$ by Pr_t/Pr, Churchill [11] re-expressed Eq. (5) in terms of the eddy conductivity and thereby in terms of Pr_t and $(\overline{u'v'})^{++}$ by means of the following series of steps:

$$\frac{j}{j_{w1}} = \left(1+\frac{k_t}{k}\right)\frac{dT^+}{dr^+} = \left[1+\left(\frac{k_t}{c\mu_t}\right)\left(\frac{c\mu}{k}\right)\left(\frac{\mu_t}{\mu}\right)\right]\frac{dT^+}{dr^+}$$
$$= \left[1+\left(\frac{Pr}{Pr_t}\right)\left(\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)\right]\frac{dT^+}{dr^+}. \tag{7}$$

Elimination of dT^+/dr^+ between Eq. (5) and the latter form of Eq. (7) then results in

$$\frac{Pr_t}{Pr} = \frac{(\overline{u'v'})^{++}\left(1-(\overline{T'v'})^{++}\right)}{(\overline{T'v'})^{++}\left(1-(\overline{u'v'})^{++}\right)}. \tag{8}$$

Eq. (8), which is exact and nominally applicable for all geometries and thermal boundary conditions, is a surprising result in that Pr_t is seen to be independent of its heuristic diffusional origin as $c\mu_t/k_t$, and instead simply an expression for the ratio of the fraction of the radial transport of momentum (the local shear stress) due to turbulence to that due to molecular motion, divided by the corresponding ratio for the transport of energy (the local heat flux density). Eq. (8) is superficially misleading in that Pr_t approaches a limiting value as Pr increases, and becomes unbounded as Pr decreases owing to the variation of $(\overline{T'v'})^{++}$. The principal merits of Pr_t vis-à-vis $(\overline{T'v'})^{++}$ are (1) its constrained behavior in the range of ordinary fluids such as air and water, (2) its minimal dependence on location and the rate of flow by virtue of their representation by $(\overline{u'v'})^{++}$, and (3) the possibility of its theoretical prediction by *renormalization group theory*. (See, for example, [12].)

Owing to the singular behavior of $(\overline{u'v'})^{++}$ in an annulus, it is convenient to re-express Eq. (8) in terms of $(\overline{u'v'})^+$ as follows:

$$\frac{Pr}{Pr_t} = \frac{(\overline{u'v'})^+[1-(\overline{T'v'})^{++}]}{(\overline{T'v'})^{++}[(\tau/\tau_{w1})-(\overline{u'v'})^+]}. \tag{9}$$

The correlating equations devised by Kaneda [6] for $(\overline{u'v'})^+$ avoid the possibility of singular behavior in Eq. (9). Substituting for $(\overline{T'v'})^{++}$ in Eq. (5) from Eq. (9) results, after rearrangement, in

$$\frac{j}{j_{w1}} = \frac{1}{a_1^+}\left[1+\left(\frac{Pr}{Pr_t}\right)\left(\frac{(\overline{u'v'})^+}{\tau/\tau_{w1}-(\overline{u'v'})^+}\right)\right]\frac{dT^+}{dR} \tag{10}$$

Here $a_1^+ = a(\tau_{w1}\rho)^{1/2}/\mu$. For uniform heating of the inner wall and no heat flux through the outer wall of the annulus it may be shown that $\partial T/\partial z = \partial T_m/\partial z$. Eq. (6) then reduces to

$$\frac{j}{j_{w1}} = \frac{1}{R[(a_2/a_1)^2-1]}\int_{R^2}^{(a_2/a_1)^2}\left(\frac{u}{u_m}\right) dR^2. \tag{11}$$

Eq. (10) may be integrated formally to obtain

$$T^+ = a_1^+\int_1^R \frac{(j/j_{w1})\,dR}{1+\left(\frac{Pr}{Pr_t}\right)\left(\frac{(\overline{u'v'})^+}{(\tau/\tau_{w1})-(\overline{u'v'})^+}\right)}. \tag{12}$$

The mixed-mean temperature, and thereby the Nusselt number can in turn be determined by integration of T^+, weighted by u/u_m over the channel, that is

$$\frac{2(a_2^+-a_1^+)}{Nu} \equiv T_m^+$$
$$\equiv \frac{1}{(a_2/a_1)^2-1}\int_1^{(a_2/a_1)^2} T^+\left(\frac{u}{u_m}\right) dR^2. \tag{13}$$

Here a characteristic length of $2(a_2 - a_1)$ is implied for Nu. Substitution in Eq. (13) of T^+ from Eq. (12) results in the double integral

$$T_m^+ = \frac{a_1^+}{[(a_2/a_1)^2 - 1]} \times \int_1^{(a_2/a_1)^2} \left[\int_1^R \frac{(j/j_{w1})\,dR}{1 + \left(\frac{Pr}{Pr_t}\right)\left(\frac{\overline{(u'v')}^+}{(\tau/\tau_{w1}) - \overline{(u'v')}^+}\right)} \right] \left(\frac{u}{u_m}\right) dR^2. \tag{14}$$

If j/j_{w1} from Eq. (11) were substituted in Eq. (14) it would become a triple integral. Although this double or triple integral can be solved directly by quadrature, the simultaneous step-wise solution of a finite-difference formulation of Eq. (10) together with finite-difference formulations of the differential forms of Eqs. (11) and (13), is more efficient computationally.. The primary value of these integral formulations is the revelation that general expressions for T, T_m^+, and Nu in terms of the basic local variables can be placed in such explicit and compact forms. In the instances of round tubes and parallel-plate channels, the replacement of j/j_{w1} by $\gamma = (j/j_{w1})/(\tau/\tau_{w1}) - 1$ in the equivalent of Eq. (14) proved very useful in that the possibility of integrating the double integral by parts then became evident, and the resulting reduced forms lead to very simple asymptotic expressions for $Pr = 0$ and $Pr = Pr_t$. These latter substitutions are not so convenient for annuli because of the more complex dependence of on r, and, in particular, its change of sign at $r = a_0$, and they are not therefore introduced here.

4. An asymptotic expression for Nu_∞

In so far as Pr_t approaches a fixed value, here designated as Pr_t^*, at the heated surface, the asymptote for $Pr \to \infty$ in fully developed turbulent convection is

$$Nu = 0.07374\,Re\left(\frac{f_{w1}}{2}\right)^{1/2}\left(\frac{Pr}{Pr_t^*}\right)^{1/3} = 0.07374\,Re\left(\frac{f_{wm}}{2}\right)^{1/2}\left(\frac{Pr}{Pr_t^*}\right)^{1/3}\left(\frac{\tau_{w1}}{\tau_{wm}}\right)^{1/2}. \tag{15}$$

The derivation of Eq. (15), which is free of empiricism but incorporates a coefficient determined from direct numerical simulations, may be found in [13,14], as well as in lesser detail elsewhere. Here $f_{w1} = 2\tau_{w1}/\rho u_m^2 = 2/(u_m^+)_{w1}^2$. Eq. (15), which is independent of the choice of a characteristic length, has been found experimentally to be applicable for all shear flows. It may not be applicable for $Pr > 100$ because of the possible failure of Pr_t to approach a fixed value at the surface (see, [15]), but, in any event, this is not be a serious limitation in a practical sense since Pr is less than 100 for all ordinary fluids.

5. Numerical calculations and required inputs

Numerical calculations for T^+ and T_m^+ (and thereby for Nu) were carried out for $Pr = 0$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 0.3, 0.7, 0.8673, 1, 3, 10, 100, 1000, and 10,000; for $a_1/a_2 = 0.01$, 0.05, 0.1, 0.2, 0.5, 0.8, 0.9, 0.95, 0.99, and 0.999; and for a wide range of values of $(a_2^+ - a_1^+)_{w1}$ extending from 150, the lowest possible value for fully turbulent flow, up to 10^6, which is beyond that for any practical application.

As mentioned in Section 3, the numerical computations were carried out using step-wise integration of differential formulations. In order to perform these numerical integrations for T^+ and T_m^+, it is necessary to have analytical expressions or numerical values for τ/τ_{w1}, a_0/a_1, a_{max}/a_1, $\overline{(u'v')}^+$, u^+, u_m^+, and Pr/Pr_t. Such expressions and representative tabulated values for a wide range of conditions were presented in Part I.

The only remaining requirement is an expression for Pr_t. The very old but unique set of experimental data of Abbrecht and Churchill [16] for Pr_t in a developing temperature field indicate unambiguously that this quantity is independent of geometry and the thermal boundary condition on the surface(s) and a function only of Pr and $\overline{(u'v')}^{++}$, at least in the turbulent core. The latest experimental data and theoretical analyses further suggest that the dependence of Pr_t on $\overline{(u'v')}^{++}$ is of second order. (See [12].) Fortuitously, Nu, as shown by the test calculations of Yu et al. [17] with several different expressions for Pr_t as a function of Pr and $\overline{(u'v')}^{++}$, is very insensitive to this choice. In view of all these considerations, the following numerically modified empirical equation of Jischa and Rieke [18], which is among the simplest of those that have been proposed, was utilized in the current work:

$$Pr_t = 0.85 + \frac{0.015}{Pr}. \tag{16}$$

Eq. (16) implies that Pr_t is independent of $\overline{(u'v')}^{++}$ and nearly invariant with respect to Pr for all ordinary fluids such as air, water, and hydrocarbons.

6. Calculated results

The computed values of $Nu_0 \equiv Nu\{Pr = 0\}$ and $Nu_1 = Nu\{Pr = Pr_t\}$, summarized in Tables 1 and 2, respectively, as a function of the chosen regular sequence of values of a_1/a_2, $(a_2^+ - a_1^+)_{w1}$ and Pr. These values of Nu_0 and Nu_1 are based on a characteristic length of $2(a_2 - a_2)$ in the hope of minimizing the explicit dependence on a_1/a_2. The corresponding values of $(u_m^+)_{wm} = (\tau_{wm}/\tau_{w1})^{1/2}(u_m^+)_{w1}$, as determined from the directly computed and tabulated values of $(u_m^+)_{w1}$ by Kaneda et al. [6] are listed in Table 3. The blank spaces in Tables 1–3 represent conditions for which the achieve-

Table 1
Computed values of Nu_0

a_1/a_2	0.01	0.05	0.1	0.2	0.5	0.8	0.9	0.95	0.99	0.999
τ_{wm}/τ_{w1}	0.4239	0.6382	0.7425	0.8420	0.9479	0.9860	0.9937	0.9970	0.9994	0.9999
$(a_2^+ - a_1^+)_{w1}$										
150		17.12	11.48	8.294	6.193	5.729	5.659	5.632	5.614	5.610
500	52.27	17.21	11.54	8.333	6.284	5.847	5.784	5.761	5.745	5.742
800	52.32	17.20	11.55	8.342	6.303	5.874	5.813	5.790	5.775	5.772
1000	52.28	17.20	11.55	8.345	6.310	5.884	5.824	5.801	5.787	5.784
2000	52.25	17.20	11.55	8.350	6.325	5.907	5.849	5.828	5.814	5.811
5000		17.19	11.54	8.351	6.337	5.927	5.871	5.851	5.838	5.835
10,000		17.18	11.54	8.350	6.343	5.937	5.883	5.864	5.851	5.848
20,000			11.53	8.348	6.348	5.946	5.893	5.874	5.862	5.860
50,000			11.52	8.347	6.354	5.958	5.906	5.888	5.876	5.874
100,000			11.52	8.348	6.360	5.967	5.917	5.899	5.888	5.885
200,000				8.351	6.369	5.980	5.931	5.913	5.902	5.900
500,000				8.364	6.391	6.009	5.961	5.945	5.934	5.932

Table 2
Computed values of Nu_1

a_1/a_2	0.01	0.05	0.1	0.2	0.5	0.8	0.9	0.95	0.99	0.999
τ_{wm}/τ_{w1}	0.4239	0.6382	0.7425	0.8420	0.9479	0.9860	0.9937	0.9970	0.9994	0.9999
$(a_2^+ - a_1^+)_{w1}$										
150		33.80	22.00	13.91	13.81	14.45	14.65	14.75	14.82	14.84
500	98.33	46.55	42.86	40.87	40.72	41.66	41.97	42.12	42.24	42.27
800	101.2	72.24	66.80	63.29	62.53	63.74	64.15	64.35	64.51	64.54
1000	116.0	89.64	82.56	77.89	76.69	78.07	78.56	78.80	78.98	79.02
2000	222.6	177.4	159.6	148.4	144.9	147.1	147.9	148.3	148.6	148.7
5000		445.9	381.7	348.1	337.1	341.7	343.4	344.3	345.0	345.1
10,000		837.3	740.5	665.2	640.7	648.8	652.1	653.6	654.7	655.0
20,000			1443	1275	1221	1236	1241	1244	1246	1247
50,000			3531	3035	2880	2909	2922	2928	2933	2934
100,000			6683	5892	5539	5585	5608	5619	5628	5628
200,000				11,560	10,720	10,780	10,820	10,830	10,850	10,850
500,000				29,160	26,150	26,080	26,130	26,150	26,180	26,160

Table 3
Derived values of $(u_m^+)_{wm}$

a_1/a_2	0.01	0.05	0.1	0.2	0.5	0.8	0.9	0.95	0.99	0.999
τ_{w1}/τ_{wm}	2.359	1.567	1.347	1.188	1.055	1.014	1.006	1.003	1.001	1.000
$(a_2^+ - a_1^+)_{w1}$										
150		12.07	12.28	12.52	12.58	12.57	12.57	12.56	12.56	12.56
500	15.80	16.33	16.49	16.62	16.75	16.81	16.82	16.83	16.83	16.83
800	17.21	17.66	17.81	17.94	18.07	18.14	18.16	18.17	18.17	18.18
1000	17.81	18.26	18.41	18.53	18.67	18.74	18.76	18.77	18.77	18.78
2000	19.61	20.02	20.16	20.27	20.41	20.50	20.52	20.53	20.54	20.54
5000		22.23	22.35	22.46	22.60	22.69	22.72	22.74	22.75	22.75
10,000		23.83	23.95	24.06	24.21	24.31	24.34	24.35	24.37	24.37
20,000			25.53	25.64	25.79	25.90	25.93	25.95	25.96	25.97
50,000			27.58	27.68	27.85	27.97	28.00	28.02	28.04	28.04
100,000			29.08	29.19	29.36	29.49	29.53	29.55	29.57	29.57
200,000				30.61	30.80	30.95	30.99	31.02	31.03	31.04
500,000				32.25	32.48	32.66	32.71	32.74	32.76	32.76

ment of fully developed turbulence or convergence of the computations for either the velocity field as indicated by the mismatching of the peak in the velocity or the equivalent in the temperature field was uncertain or for which some other anomaly such as a value of the Reynolds number below the presumed minimum for fully developed turbulent flow was observed. Values of τ_{w1}/τ_{wm} are included in Tables 1 and 2 for convenience because these latter quantities are utilized directly in the correlating expressions for Nu_0 and Nu_1 that are developed here and in more general form in Part **IV** for values of a_1/a_2 and $(a_2^+ - a_1^+)_{w1}$ intermediate to those listed.

The tabulations of Nu_0 and Nu_1 provide an economical but sufficient representation of the computations for heat transfer since, as shown in the next section, they can be used to calculate Nu, with virtually no added error, from simple algebraic expressions for all values of Pr_t/Pr.

The values of $(u_m^+)_{wm}$ in Table 3 may be observed to be virtually independent of a_1/a_2. On this basis Kaneda et al. [6] suggested their approximate representation by

$$(u_m^+)_{wm} = \left(\frac{2}{f_{wm}}\right)^{1/2} = 3.2 + \frac{1}{0.436}\ln\{Re(f_{wm}/8)^{1/2}\} - \frac{275}{Re(f_{wm}/8)^{1/2}}. \tag{17}$$

Eq. (17) differs from the prior expression of Yu et al. [17] for round tubes and that of Danov et al. [19] for parallel-plate channels only in the last term. This term is an arbitrary approximation for the differing terms for the decrease in the mean velocity due to the departure from semi-logarithmic behavior in the viscous and buffer layers. Values of u_m^+ and f are here normalized in Eq. (17) in terms of τ_{wm} rather than in terms of τ_{w1}, not only because the of the virtual elimination of a_1/a_2 as a parameter, because also because of the more direct practical interest ensuing from the following relationship between τ_{wm} and the pressure gradient:

$$\left(-\frac{dP}{dx}\right) = \frac{2(a_1\tau_{w1} + a_2\tau_{w1})}{(a_2^2 - a_1^2)} = \frac{2\tau_m}{a_2 - a_1}. \tag{18}$$

The dichotomy in the normalization of u_m^+, f, and $a_2^+ - a_1^+$ in terms of τ_{w1} in some instances and in terms of τ_{wm} in others is essentially unavoidable because u_m was not known in advance, precluding the choice of a series of regular values of Re or $(a_2^+ - a_1^+)_{wm}$ for the numerical integrations, and because the behavior of the flow near the wall depends directly on τ_{w1} rather than on τ_{wm}.

7. Representation of the calculated values

An expression for representation of computed values of Nu for a round tube was devised by Churchill et al. [20] on the basis of the analogy of Reichardt [21], as assembled in compact form and corrected by Churchill [14], namely

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left(1 - \left(\frac{Pr_t}{Pr}\right)\right)\frac{1}{Nu_\infty}. \tag{19}$$

Eq. (19), which is free of any explicit empiricism, is obviously valid only for $Pr_t/Pr \leqslant 1$, but its analog for $Pr_t/Pr \geqslant 1$ may readily be inferred on the basis of symmetry. These two expressions proved to be very successful for representation of computed values of Nu for both round tubes and parallel-plate channels, and, as speculated in advance, for several different thermal boundary conditions. Indeed, the deviations were barely distinguishable visually in plots in logarithmic coordinates. However, Churchill and Zajic [22] determined by means of more critical comparisons that the deviations were actually as great as 10% for Pr of the order of 10, and as great as 30% for Pr of the order of 0.01. They concluded that these deviations were a consequence of the idealizations made by Reichardt [21] in order to be able to integrate the combined differential momentum and energy balance analytically. When an alternative analogy, derived earlier by Churchill [14] but not at first recognized as necessarily an improvement for want of sufficiently accurate computed values or experimental data to provide a critical comparison with that of Reichardt, was substituted these deviations vanished for all practical purposes. The revised expression for $Pr_t/Pr \leqslant 1$ has the form:

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left(1 - \left(\frac{Pr_t}{Pr}\right)^{2/3}\right)\frac{1}{Nu_\infty}. \tag{20}$$

Insofar as the effect of the variation in Pr_t with Pr over the range of applicability of Eq. (20) is negligible, Eq. (20) can be simplified to

$$\frac{1}{Nu} = \frac{Pr_t}{Pr}\left(\frac{1}{Nu_1} + \left[\left(\frac{Pr}{Pr_t}\right)^{2/3} - 1\right]\frac{1}{Nu_\infty^1}\right). \tag{21}$$

Here, $Nu_\infty^1 \equiv Nu_\infty\{Pr = Pr_t\} = 0.07343\,Re(f_{w1}/2)^{1/2} = 0.07343\,Re(f_{wm}/2)^{1/2}(\tau_{w1}/\tau_{wm})^{1/2}$. The simplification consists of the elimination of Pr_t^*, which a parameter of Nu_∞ but not of Nu_∞^1. The resulting maximum numerical error is less than 0.7%.

The analogue of Eq. (21) for $Pr_t/Pr \geqslant 1$ is

$$\frac{Nu_1 - Nu_0}{Nu_1 - Nu} = 1 + \frac{(Pr_t/Pr)^{1/8}(Nu_1 - Nu_0)Nu_\infty^1}{\left(\frac{Pr_t}{Pr} - 1\right)\left(Nu_\infty^1 - \frac{2}{3}Nu_1\right)Nu_1} \tag{22}$$

The exponent of 1/8 in Eq. (22) is the only empirical element in any of these expressions, and the impact of the term $(Pr_t/Pr)^{1/8}$ on Nu is quite limited owing to the dominance of the term $(Pr_t/Pr - 1)$.

As may be seen in Fig. 1, Eqs. (21) and (22), together with Nu_0 and Nu_1 from Tables 1 and 2, and Nu_∞ and Nu_∞^1 from Eq. (15), represent the dependence of the computed values of Nu on Pr/Pr_t almost perfectly for all values of $(a_2^+ - a_1^+)_{w1}$ and all values of a_1/a_2, except for the range of $Pr_t/Pr = 10^3$–10^4 for $a_1/a_2 = 0.01$, which, in view of the overall success of Eq. (22), may indicate mild error in these particular computed values rather than in the correlative expression. The overall representation of the computed values of Nu for annuli by Eqs. (21) and (22) is not only a great success in its own right, but in view of this third geometry and the unsymmetrical thermal boundary condition, it appears to provide the final confirmation of the conjecture of Churchill et al. [20] of the complete generality of these two expressions in both of these respects.

For direct comparisons with experimental or prior computed values of Nu it is necessary to interpolate the values of Nu_0 and Nu_1 in Tables 1 and 2 for intermediate values of a_1/a_2 and $(a_2^+ - a_1^+)_{w1}$. Generalized, empirical correlating equations for such interpolations for all geometries and conditions are presented in Part IV. For the more constrained conditions encompassed by the prior experimental work, namely air, water, and mercury at $Re < 250{,}000$, the following simple expressions proved to be adequate:

$$Nu_0 = \frac{5.89\left(1 + \frac{0.0703}{(a_1/a_2)^2}\right)^{1/3}}{1 + \frac{0.8}{(u_m^+)_{wm}}\left(\frac{a_1}{a_2}\right)} \tag{23}$$

and

$$Nu_1 = \frac{(1 + 0.288(a_1/a_2)^{0.28})Re(\tau_{w1}/\tau_{wm})}{1.288(u_m^+)_{wm}^2\left(1 + \left(\frac{53.3}{(u_m^+)_{wm}}\right)^3\right)^{1/10.6}}. \tag{24}$$

Eqs. (23) and (24) represent the values in Tables 1 and 2 almost exactly for $Re < 100{,}000$ and $a_1/a_2 \geqslant 0.1$, and reasonably well for $100{,}000 < Re < 240{,}000$ and $0.01 \leqslant a_1/a_2 \leqslant 0.1$.

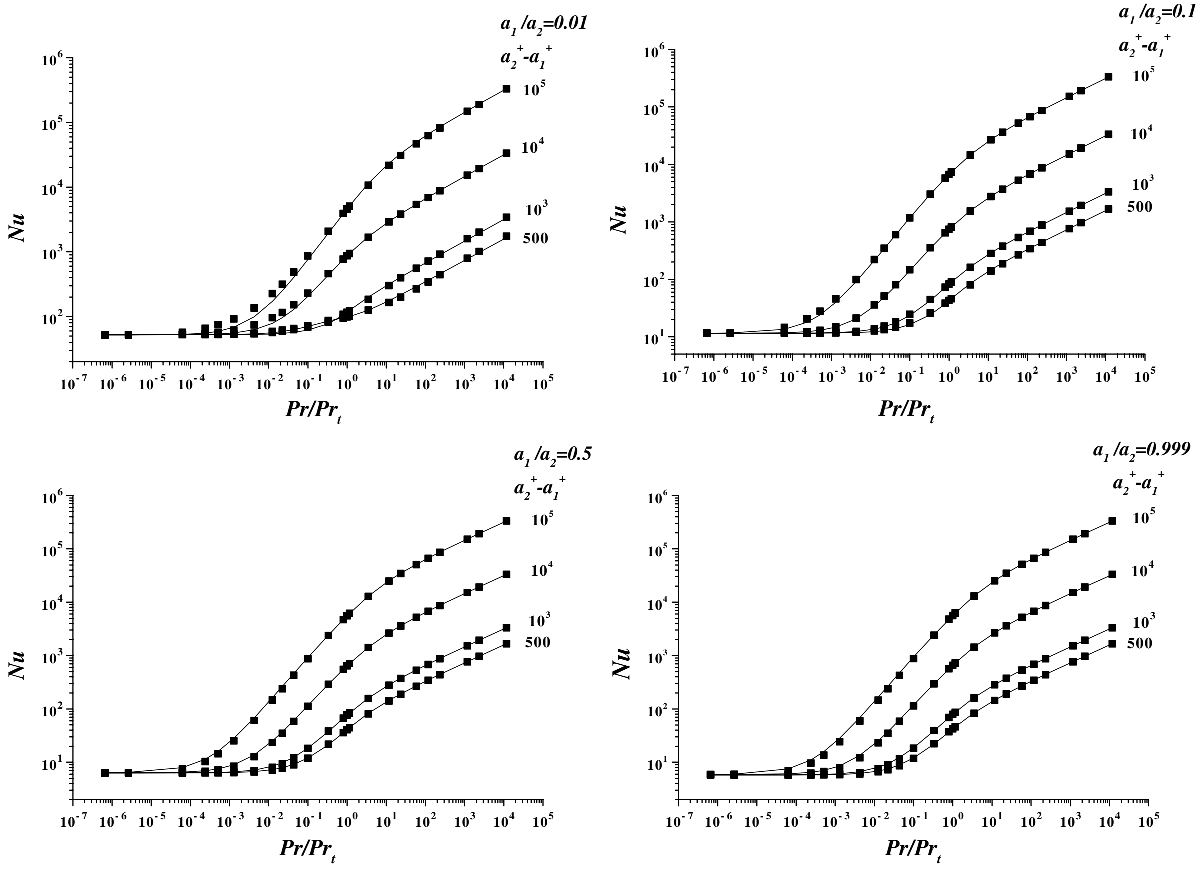

Fig. 1. Representation of computed values of Nu with predictions of Eqs. (21) and (22).

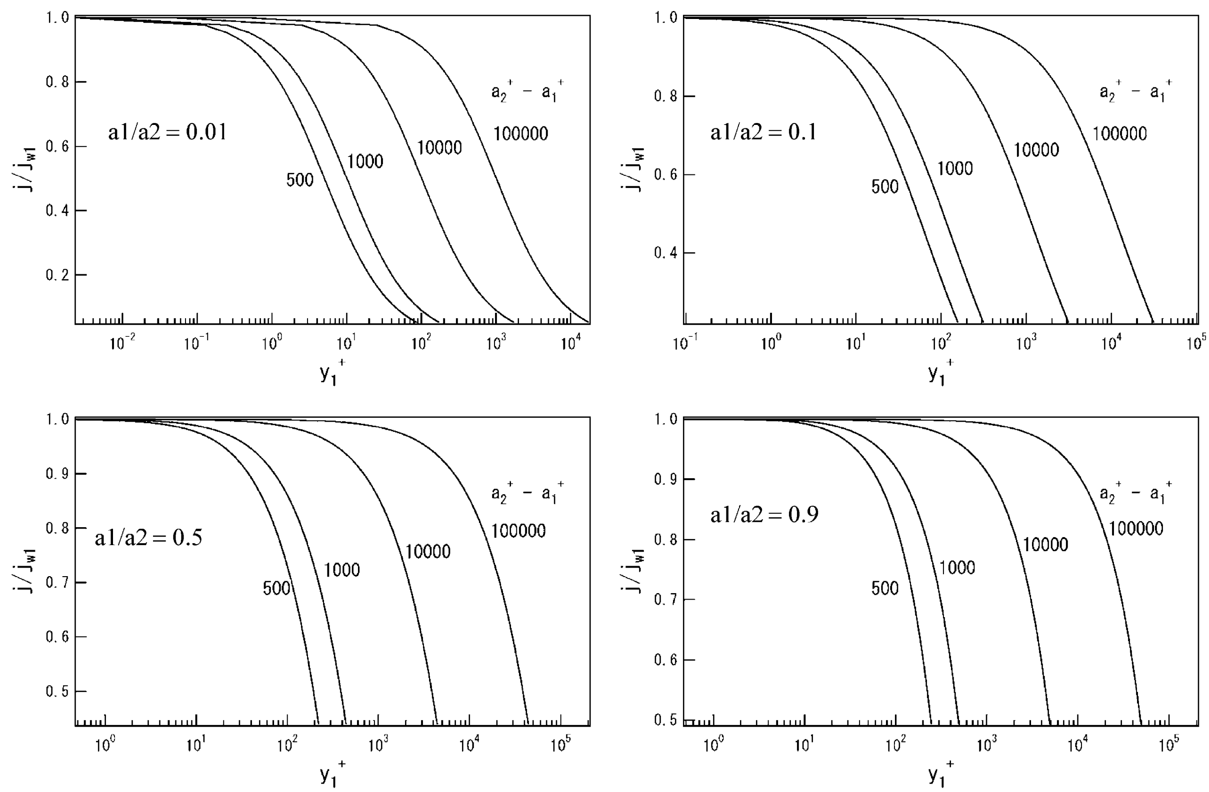

Fig. 2. Computed heat flux density ratio as a function of distance from the wall.

The plots in Fig. 2 of j/j_{w1} versus the fractional distance across the channel indicate that this quantity approaches zero in the inner region and is thereby of very small magnitude in the outer region.

8. Comparisons with prior computed and experimental values

Experimental data for the time-averaged velocity distribution were utilized in Part 1 to provide a critical test of the numerically computed values and thereby of the empiricisms in the modeling of the flow, namely the correlating equations for a_0, $a_{\max}$, and $(\overline{u'v'})^+$. Experimental data of equivalent extent and reliability do not exist for the time-averaged temperature distribution. Therefore comparisons of the results of the thermal computations with experimental data are limited to the Nusselt number. Although many sets of experimental data for Nu were found, they are generally very old and subject to large temperature differences, and thereby to significant variations in k and μ. Very small inner diameters, and especially those corresponding to electrically heated wires, are subject to misalignment and/or mislocation, the slightest degree of which distorts the flow and thereby affects the rate of convection.

Because Nu is a function of many independent variables and parameters, including, Re, Pr, a_1/a_2, T_m^+, and T_w/T_m, and because the experimental data are generally for irregularly spaced values of these quantities, parametric plots are not feasible for comparison with the predictions. Instead the comparisons were made in terms of plots of Nu_{exp} from Carpenter et al. [23], McMillan and Larson [24], Dufinescz and Marcus [25], Monrad and Pelton [26], T.E.M.A. [27], and Miller et al. [28] versus Nu_{pred} for water ($2 \leqslant Pr \leqslant 10$), from Leung [1], Monrad and Pelton [26], T.E.M.A. [27], Petukhov and Roizen [29], Roberts and Barrow [30], Quarmby [31], Vilemas et al. [32], and Zerban [33] for air ($Pr \cong 0.70$), and from Trefethan [34] for mercury ($0.01 \leqslant Pr \leqslant 0.03$), as shown in Figs. 3–5, respectively, with the sources and parameters of each set or segment or data identified by coding.

Except in the few instances in which sufficient information was given to permit calculation of the average value of the wall and mixed-mean temperatures, the values of Nu, Re, and Pr given by the authors were necessarily utilized in the comparisons with the predicted values of Nu. When sufficient details were given, the value of Pr for water was evaluated at the average of

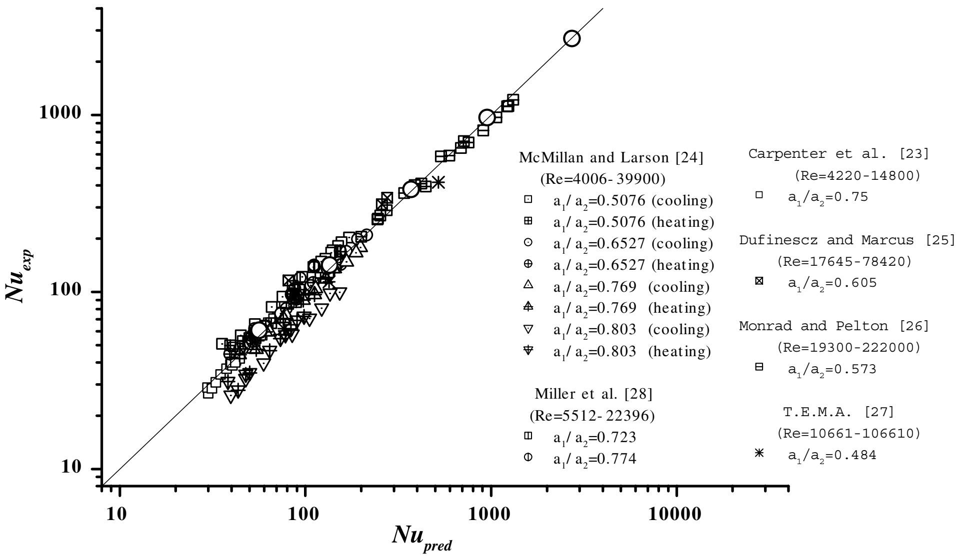

Fig. 3. Comparison of predicted values with experimental data for water and predictions of Kays and Leung [3].

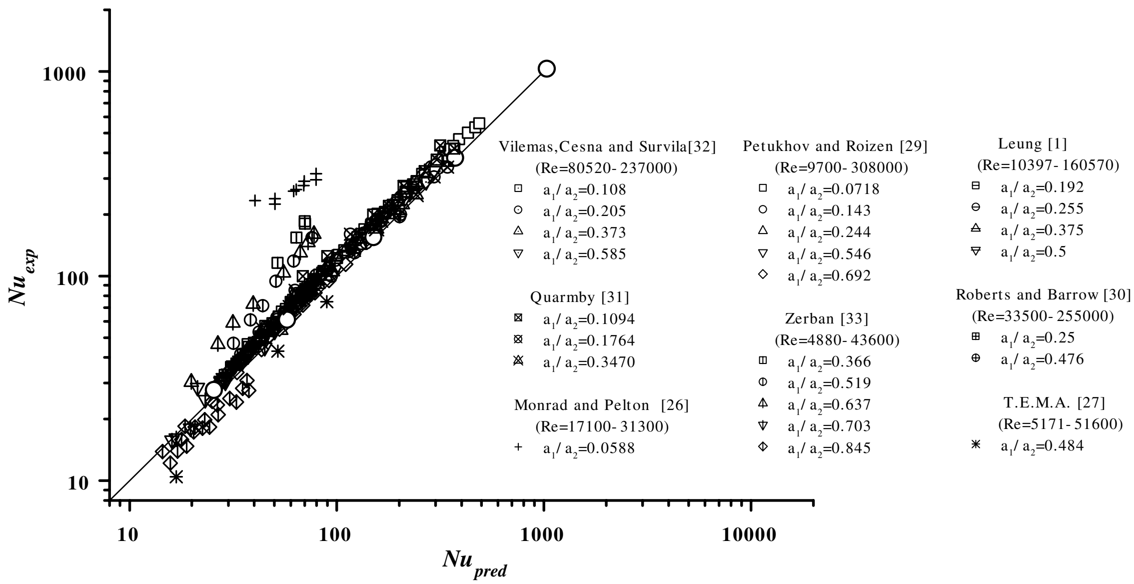

Fig. 4. Comparison of predicted values with experimental data for air and predictions of Kays and Leung [3].

the wall and mixed-mean temperatures for each experimental point. In the absence of specified temperatures for the individual data points for mercury, a mean value of $Pr = 0.02$ was used. For data given only in the composite forms such as $Nu/Re(f/2)^{1/2}Pr^n$, values of f, Pr and n were necessarily estimated. For experiments in which z/D was varied, the apparent asymptotic value of Nu was estimated. Vilemas et al. [32] are the only experimenters to have followed the commendable practice of extrapolating their values of Nu to zero temperature-difference. The several sets of data in which a heated wire was used for the inner surface were excluded from Figs. 3 and 4 both because they were for values of a_1/a_2 far less than those of the numerical integrations, and because they were wildly scattered, particularly on the high side. The data for heated rods and for moderate aspect ratios did not demonstrate such extreme behavior and were retained.

The predictions of Eq. (21) and its components appear in Fig. 3 to be in good agreement on the mean with

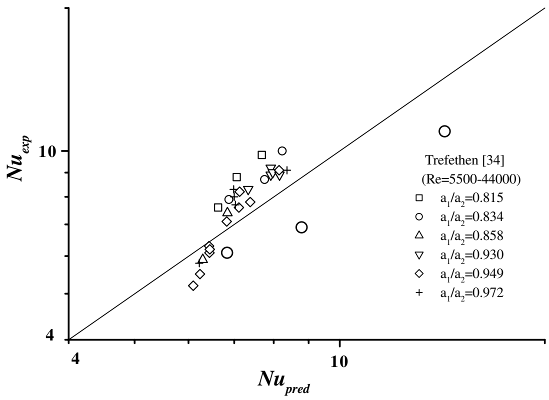

Fig. 5. Comparison of predicted values with experimental data for mercury and predictions of Kays and Leung [3].

the experimental data for water, and it seems reasonable to attribute the deviations to physical property, entrance, and alignment effects as well as to experimental or predictive error. The experiments of McMillan and Larson [24] for heating and cooling of water are identified by separate symbols. Their values of *Nu* for cooling may be observed to be slightly higher on the mean than those for heating, and this effect is confirmed more definitively by direct comparison of the numerical values of *Nu* for the same values of a_1/a_2 and approximately the same values of *Re*. The values of *Nu* plotted in Figs. 3 and 4 for all other investigators are for heating of the fluid.

As shown in Fig. 4, the deviations of the experimental data for air are somewhat greater than those in Fig. 3 for water, and those sets of data that might be presumed to be most accurate deviate somewhat consistently on the high side. No simple explanation has been found for this latter, seemingly coherent difference. It may be inferred from the experimental results of McMillan and Larson for water that heating air would produce positive deviations since the effect of temperature on the viscosity is opposite to that for water. Also, the effects of buoyancy and disturbances, both of which would be expected to increase *Nu*, might be expected to be greater for air, but a more quantitative explanation must await experimental and/or computations focused on these effects.

The single set of experimental data for mercury may be observed in Fig. 5 to differ more greatly than those for water and air on the mean but such deviations are perhaps to be expected in consideration of the inherent experimental difficulties with that fluid. It is possible that the negative deviations are a result of the failure to attain fully turbulent flow.

The computed values of Kays and Leung [3] for parallel plates are represented in Figs. 3–5 by large circles. Those for water are arbitraily based on $Pr = 3.0$ and those for mercury on $Pr = 0.03$, but such approximations would be expected to have a negligible effect in this graphical form because the same values were used in both sets of predictions. The predicted values of Kays and Leung are seen to differ negligibly from the predictions herein for water. They are slightly higher on the mean than the present predictions for air but nevertheless much closer to the latter than to the overlying band of experimental data. They are significantly lower than either the present predictions or the experimental data for mercury. A more direct and critical comparison of the earlier and present predictions in tabular form generally confirms these visual observations, and in addition indicates a more rapid increase in *Nu* as a_1/a_2 decreases. Such tabular comparisons are not presented here because the numerical and functional differences in *Nu* are simply an artifact of the different models used for the turbulent shear and the turbulent Prandtl number and not particularly meaningful in themselves. The differences in the models themselves are discussed in the next section.

9. Evaluation and interpretation

It is difficult to identify the causes of the large differences between the various sets of experimental data for *Nu* for roughly the same conditions because of incomplete quantitative characterizations by the experimenters. A need clearly exists for new, more accurate and more extensive data for turbulent convection in annuli, particularly for small temperature-differences, in order to eliminate physical property variation as a variable, or for a series of fixed temperature-differences, in order to define such effects. Data for liquids other than water and mercury are also needed. In the interim until such data are obtained, confidence in the generality and accuracy of the computed values must depend to a considerable degree on the following assessment of their theoretical credentials.

The new computed values of *Nu* are subject to errors of two types—idealizations in the differential model and in the discretization in the numerical integration. The convergence of the numerical calculations as the grid-size was reduced and the comparative use of more than one numerical algorithm for integration would appear to eliminate effectively the latter source of error. The error due to the idealizations in the model itself is more difficult to assess. The excellent agreement of prior computed values of *Nu* by Heng et al. [8] and Yu et al. [17] for round tubes, and by Danov et al. [19] for parallel-plate channels with experimental data appears to validate the general model. Even more importantly, the excellent agreement of the predicted and measured values of the time-averaged velocity distribution and the friction factor for annuli, as reported by Kaneda et al. [6] in Part I of this investigation, suggests that the correlative and predictive equations for a_0, a_{max}, $(\overline{u'v'})^+$, u^+, and u_m^+ are not a significant source of uncertainty in the modeling of T^+ and Nu for annuli. Also, because Eqs. (11)–(13) are exact formulations, the only additional source of uncertainty relative to that for flow arises from the expression for Pr_t. Unfortunately, as recently noted by Kays [35] and Churchill [14], this quantity is still subject to considerable uncertainty, both functionally and numerically. On the other hand, as mentioned in Section 5, the uncertainty in Nu resulting from this uncertainty Pr_t is much less. The principal uncertainty in Pr_t is for very small and very large values of Pr. For very small values of Pr, molecular transport is dominant over turbulent transport and hence Pr rather than Pr_t is the controlling factor. Also, no ordinary fluids, even including liquid metals, have Prandtl numbers less than about 0.01. At the other extreme, no ordinary fluids have Prandtl numbers greater than about 100, and the effect of the limited uncertainty in the value of Pr_t on Nu for $Pr > 0.8673$ is reduced by the 1/3-power dependence of Nu_1 on $(Pr/Pr^{1/3})$.

Owing to the essentially exact expression used for $(\overline{u'v'})^+$ as compared to the inaccurate values used in the past for the eddy viscosity and the mixing-length, and in many instances owing to the less accurate expressions used for the time-averaged velocity and heat flux density distributions as well, the new computed values of Nu are presumed to be superior in numerical and functional accuracy to all those of the past, with the exception of the limited ones for very small Re obtained by DNS. Even so, the quantitative improvement in the prediction of Nu is quite small, as is demonstrated by the comparisons in Figs. 3–5 of the current predictions with those of Kays and Leung [3], which were based on (1) the eddy diffusivity, a quantity that is now known to be fundamentally unsound in an annulus; (2) a velocity, distribution which is incoherent with the the expression used for the eddy viscosity; and (3) a fourth-power dependence of the turbulent shear stress on y^+ near the wall, whereas a third-power dependence has recently been confirmed beyond any question by DNS. There are two explanations for the limited improvement in a numerical sense. One is the insensitivity of Nu to the expressions used for the eddy diffusivity and the velocity distribution because of the smoothing resulting from the double integration to obtain T_m^+. The other is that the arbitrary coefficients of the two models are ultimately based on essentially the same experimental values of the velocity distribution.

On the other hand, the improvement of the new predictions, as represented by Eqs. (21) and (22), over all prior correlating equations in the form of products of power functions is very significant in terms of both functionality and scope. These new predictive expressions incorporate a widely varying but essentially exact dependence on Pr for a complete range of that variable, and an equivalent coupled dependence on Re for all values in the turbulent regime. These new predictive expressions appear to share one feature with many of the classical power-law expressions, namely an independence from a_1/a_2 insofar as Nu and Re are expressed in terms of the *hydraulic diameter*. However this commonality is merely superficial. The new predictive expressions introduce a dependence on a_1/a_2 by virtue Nu_0 and Nu_1.

10. Numerical implementation of the new predictive expressions

The implementation of Eqs. (21) and (22) to obtain a numerical value of Nu for any value of Pr for one of the pairs of values of $(a_2^+ - a_1^+)_{w1}$ and a_1/a_2 chosen for the illustrative computations only requires the supplemental use of Eq. (16) or its equivalent for Pr_t, and the appropriate values for Nu_0 and Nu_1 from Tables 2 and 3. The corresponding value of Re is given in Table 5 of Part I [6]. This greater complexity as compared to that involved in numerical evaluations using traditional correlating equations in the form of the product of powers of Re and Pr is a small price to pay for the much better functionality and the far greater scope of the new expressions, even apart from the moderate improvement in numerical accuracy. Also, this complexity is not a factor if even a handheld calculator is utilized.

For values of $(a_2^+ - a_1^+)_{w1}$ (or Re) and a_1/a_2 intermediate to those chosen for the illustrative computations it is necessary to (1) carry out additional numerical integrations for the velocity distribution and the mixed-mean velocity or utilize the correlating equations given in Part I for the former and Eq. (17) for the latter, and (2) carry out additional numerical integrations for Nu or utilize Eqs. (23) and (24), which in turn require in addition to the value $(u_m^+)_{wm}$, a value for τ_{w1}/τ_{wm} as obtained from algebraic equations given in Part I.

11. Conclusions

The new numerically computed solutions for Nu for fully developed turbulent convection in an annulus heated uniformly on the inner wall are concluded on the basis of theoretical considerations to be more accurate numerically and particularly functionally than any prior ones. The new predictive equations, which reproduce the numerically computed values almost exactly, encompass all values of Pr, a wide range of aspect ratios, and the complete range of Re for fully developed turbulent convection. The available experimental data appear to confirm these assertions, but because of their limited scope and differences from set to set they do not provide a critical test of the new predictions.

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